Announcements

1) SVD on HW #2, problem 2 compute reduced and full

Orthogonal Projections and Gram-Schmidt

I will not say "projector"!

Book says "projector",

I say "projection".

Definition: (projection, complementary projection)

$P \in \mathbb{C}^{n \times n}$ is called a **projection** if

$$\boxed{P^2 = P}$$

Example 1:

$P = I_n$ is a projection.

Also $P = O_n$ is a projection

$P = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$ is

also a projection.

If $P^2 = P$, then

$$(I_n - P)^2 = I_n - 2P + P^2$$
$$= I_n - 2P + P$$
$$= I_n - P.$$

So $I_n - P$ is also a projection, called the Complementary projection to P.

Definition: (orthogonal projection)

A projection $P \in \mathbb{C}^{n \times n}$ is called orthogonal if $P(\mathbb{C}^n)$ and $(I_n - P)(\mathbb{C}^n)$ are $\boxed{\text{orthogonal}}$ subspaces of $\mathbb{C}^n$.

Example 2:

I_n and O_n are orthogonal projections.

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

are the orthogonal projections onto the x- and y- axes, respectively. They are complements to eachother.

$$P = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

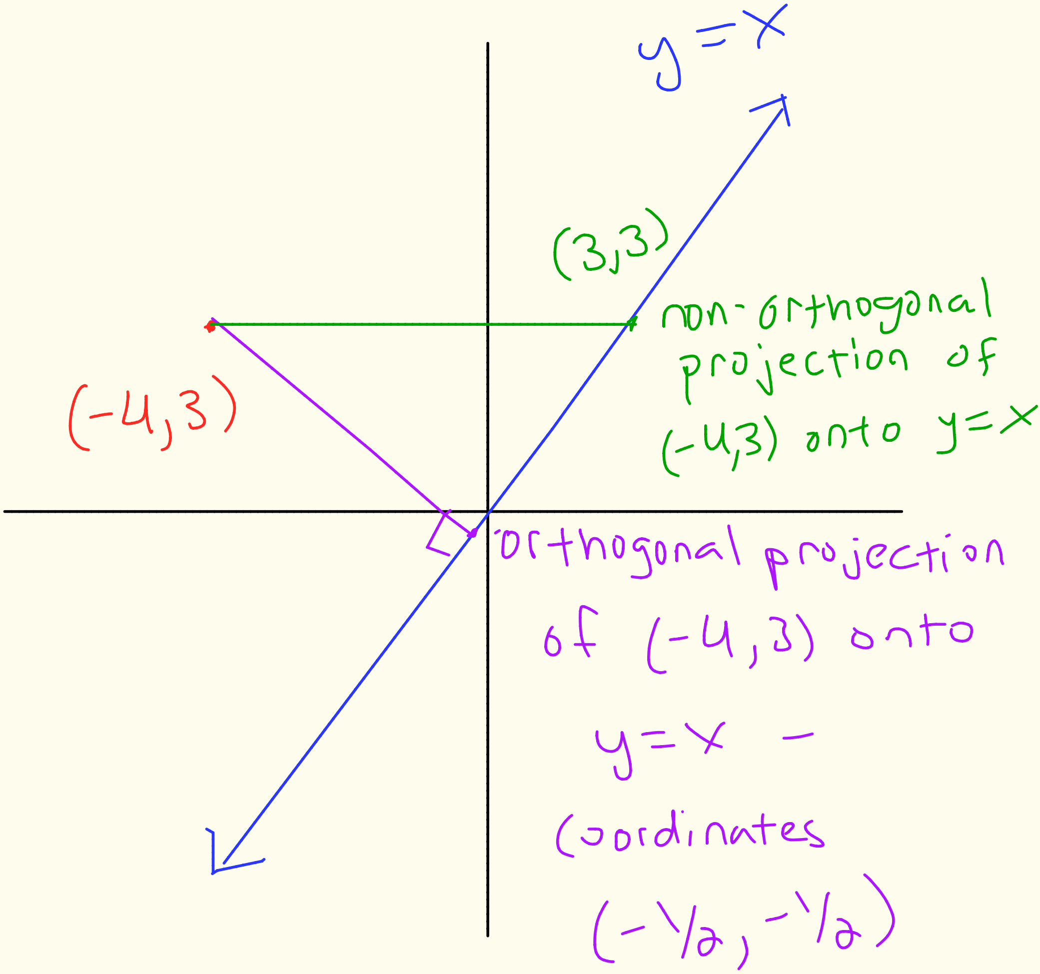
is an orthogonal projection onto the line $y = x$.

$P = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}$ is not an orthogonal projection.

See the next theorem!

The Pictures

Let K be the subspace of $\mathbb{R}^2$ given by the line $x = y$.


y = x
(3,3)
non-orthogonal projection of (-4,3) onto y=x
(-4,3)
orthogonal projection of (-4,3) onto y=x - coordinates (-1/2, -1/2)

Orthogonal projection onto $y=x$ is

$$P_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

The non-orthogonal one pictured is

$$P_2 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

Theorem: (characterization)

$P \in \mathbb{C}^n$ is a projection.

Then P is orthogonal

if and only if

$$\boxed{P = P^*}$$

proof: $\Leftarrow$ Suppose $P = P^*$.

Take $x, y \in \mathbb{C}^n$.

$$(Px)^*((I_n - P)y)$$

$$= x^* P^* (I_n - P) y$$

$$= x^* P (I_n - P) y$$

$$= x^* (P - P^2) y$$

$$= x^* (P - P) y$$

$$= 0$$

This shows

$P(\mathbb{C}^n)$ is orthogonal

to $(I_n - P)(\mathbb{C}^n)$,

which means P is

an orthogonal projection.

$\Rightarrow$) Suppose P is an

orthogonal projection

Defining property of A^*:

$$(Ax)^* y$$
$$= x^*(A^* y)$$

for all $x, y \in \mathbb{C}^n$.

Need to show:

$$(Px)^* y$$
$$= x^*(Py)$$

This will show $P = P^*$.

As a consequence of Gram-Schmidt, we can write $x, y \in \mathbb{C}^n$ as

$$x = x_0 + x_1$$

$$y = y_0 + y_1$$

with $\underline{x_0, y_0 \in P(\mathbb{C}^n)}$,

$$\underline{x_1, y_1 \in (I_n - P)(\mathbb{C}^n)}$$

Then

$$(Px)^* y$$

$$= (P(x_0 + x_1))^* (y_0 + y_1)$$

$$= x_0^* (y_0 + y_1)$$

$$= x_0^* y_0$$

since $P(\mathbb{C}^n)$ is orthogonal to $(I_n - P)(\mathbb{C}^n)$

$$x^*(Py)$$

$$= x^*(P(y_0+y_1))$$

$$= x^*(y_0)$$

$$= (x_0^* + x_1^*)y_0$$

$$= x_0^* y_0$$ Since $P(\mathbb{C}^n)$ is orthogonal to $(I_n - P)(\mathbb{C}^n)$.

This says

$$(Px)^* y = x^* (Py)$$

for all $x, y \in \mathbb{C}^n$, and so

$$P = P^*.$$

$\square$

Orthogonal Projections and Diagonalizability

If $P \in \mathbb{C}^{n\times n}$ is an orthogonal projection, then by the previous theorem, $P = P^*$, and so P is diagonalizable.

Write

$$P = QDQ^*$$

where Q is unitary in $\mathbb{C}^{n \times n}$ and $D \in \mathbb{C}^{n \times n}$ is diagonal with diagonal entries only zeros and ones.

Using permutation matrices,

arrange

$$D = \begin{bmatrix} I_k & O \\ O & O_{n-k} \end{bmatrix}$$

with $k = \text{rank}(P)$.

Letting $\hat{Q}$ equal

$Q \cdot \begin{bmatrix} I_k \\ 0 \end{bmatrix}$, we get

$$\boxed{P = \hat{Q}(\hat{Q})^*}$$

Decomposition and Rank One's

Writing an orthogonal projection $P = QDQ^*$ with $D = \begin{bmatrix} I_k & 0 \\ 0 & 0_{n-k} \end{bmatrix}$,

let $e_{11}, e_{22}, \ldots, e_{nn}$ be the diagonal matrix units in $\mathbb{C}^n$.

Then

$$P = \sum_{i=1}^{k} Q e_{i,i} Q^* \text{ is}$$

a sum of rank-one projections which are all orthogonal.